The Ontology of Number, Platonic and Postmodern Considerations

Introduction

"Is mathematics already out there?" (Da 140). This is how Paul Davies, mathematical physicist at the University of Adelaide, poses the question regarding the ontology of mathematics. Does mathematics, and number in particular, have some ethereal existence, or is it a social construct? Is mathematician Rudy Rucker correct in his Platonic view of mathematical objects as existing in a world he calls "Mindscape," or is mathematical truth merely so by convention? Considered differently, is the mathematician akin to an explorer who, when discovering mathematical truth stumbles on the "works of God," or a "profound and timeless reality," as Roger Penrose puts it? Or, is she the architect, or a part of a community of architects of these mathematical objects and relationships? In this paper I will consider the ontological nature of mathematics, presenting a classically modern view framed within the bounds of Christian orthodoxy. I will then compare this model to a postmodern mathematical ontology, considering what the two models have to offer one another.

The view that mathematical objects exist independently of us, in a reality that is not spatiotemporally located is known as *Platonism*[1]. According to this view, mathematical objects, such as numbers, sets and relations, are *observed* by the mathematician. They are eternal and unchanging. Platonism also asserts that mathematical objects are mind-independent. That is, the truth of a mathematical proposition regarding these objects is unaffected by the psyche of man. A belief

1 Platonism is much more broad in scope. I will only be considering Platonism in a mathematical context.

regarding the truth value of such a proposition does not determine its truth value in reality. This model, of course, borrows from Plato's dualistic model of reality. Here mathematical objects "transcend the physical reality that confronts our senses" (Da 141).

Some alternatives to the Platonic view suggest that mathematics is solely the construction of the mathematician. Non-Platonic ontologies have taken different forms and names. One model outside the Platonic tradition, which has grown in popularity among postmodern mathematical philosophers, is known as social constructivism. Paul Ernest has developed this paradigm which abandons all notions of absolutism in mathematics. Taking a fallabilist position, Ernest offers that mathematical statements, and even deductive logic for that matter, are a Wittgenstinian language game and cannot be separated from their social and historical contexts. Further, Ernest contends that mathematical knowledge itself changes and evolves through a dialectical and historical process. I will consider the Platonic and postmodern models in turn.

Mathematical Objects in the Mind of God

Let us now consider various Platonic ontologies of mathematics that are consistent with Christianity. I have taken much of the first model to be considered from Howell and Bradley's insightful book, *Mathematics in a Postmodern Age, A Christian Perspective*, in which C. Mezel's ontology is presented. I will also borrow Howell and Bradley's comments offered prior to their presentation of the ontology:

> The postmodernist does not argue that the modernist claim of objective mathematical truth is false. Indeed she cannot so argue, at least not directly, for that would play right into the modernist's hands, as it would itself be a claim to objective truth... Nevertheless, postmodernism gives us no reason to abandon the modernist presuppositions of objective truth and its knowability -- in particular, with regard to portions of mathematics... To this end, this chapter gives a "moderate" modernist model of the subject matter of mathematics and argues that it provides a reasonable ground for the possibility of mathematical knowledge. (HB 67)

Gödel's work provides some evidence that there exists room for error in the foundations of mathematics. That is, by Gödel's two incompleteness theorems, we cannot prove that mathematics sits on unshakably consistent grounds. That is not to say that there exists any evidence of inconsistency. Rather, in any sufficiently powerful axiomatic system, the axioms contained therein, along with deductive logic, are unable to prove that the system is consistent. Thus we enter into the debate with our eyes open, free from claims of epistemic certainty, but secure in the hope that a cogent a reasonable ontology can be offered.

For the Platonist, numbers in particular, and mathematical objects and relationships in general, are eternal and necessary realities. Traditionally, Christian thinkers have considered mathematical objects, along with other objects that we might label "abstract," as realities that exist in the mind of God. In this sense they are part of His divine Wisdom. Consider Wisdom's personification in Proverbs 8:23-31.

> The LORD brought me forth as the first of his works, before his deeds of old; I was appointed from eternity, from the beginning, before the world began. When there were no oceans, I was given birth, when there were no springs abounding with water; before the mountains were settled in place, before the hills, I was given birth, before he made the earth or its fields or any dust of the world. I was there when he set the heavens in place, when he marked out the horizon on the face of the deep, when he established the clouds above and fixed securely the foundations of the deep, when he gave the sea its boundary so the waters would not overstep his command, and when he marked out the foundations of the earth. Then I was the craftsman at his side. I was filled with delight day after day, rejoicing in his presence, rejoicing in his whole world and delighting in mankind.

The history of the natural sciences provides compelling evidence that mathematics is the language that the universe speaks. "The contention that mathematics is the key that enables the initiate to unlock cosmic secrets is as old as the subject [mathematics] itself" (Da 140). Physicist Eugene Wigner speaks of mathematics' "unreasonable effectiveness" in describing the physical universe. We see support for this thinking in the proverb above in that Wisdom is God's “craftsman.” It seems then that it was mathematics manifested in God's Wisdom that played a critical role in the creative process. If the universe speaks the language of mathematics, then it is reasonable to suppose that mathematics was part of his divine craftsmanship. Proverbs 8 implies then not only that Wisdom (and thus mathematics) was an essential element in the creative act, but also that it exists with God eternally.

The Greeks also connected Wisdom with mathematics. Plato believed that the study of mathematics focused the learner's mind on eternal and unchanging realities.

Howell and Bradley outline an ontology for number built somewhat on Augustine's doctrine of divine ideas. Howell and Bradley argue that mathematical objects have their being in the mind of God. These objects are thus eternal and necessary. Unlike Augustine, however, Howell and Bradley suggest that these objects are also part of creation. Several problems arise from this configuration. First, how can these objects be created if they are eternal? Do not created objects have a beginning? Second, creation can be viewed as not necessary. It is an act of God's will that is both gracious and free. It then follows that these created objects are not necessary and could have been otherwise.

Howell and Bradley suggest that these dilemmas result from a deistic model of creation. To the deist, created objects are given sufficient ontological momentum by God in creation and are then left to remain in being on their own. A biblical alternative that has been embraced by the church historically and that Howell and Bradley favor proposes that God not only brings the creation into being, but also sustains that being at every instant (consider Col. 1:16 and Heb. 1:3). Building on this idea is the so called

Continuous Creation or CC model. In this model, the first objection is noted above is answered, for it is possible for eternal beings to be created in that their being is always sustained by God even though there is no moment *in time* at which divine sustenance began. “Creation is thus an ongoing act, consisting in God's continuously sustaining all things” (HB 71). In this sense CC addresses the first dilemma mentioned above.

CC also affords Howell and Bradley an answer to the second objection. They argue:

> If *x* is a necessary being, then it is simply the case that God necessarily sustains *x* at every moment, that he sustains *x* in every possible world. For a non-necessary being *y*, by contrast, this is simply not the case; rather there are possible worlds and times in those worlds at which God does not sustain *y* (HB 71).

So, to Howell and Bradley, necessity is in this sense equivalent to God's eternal sustenance. For an object to be necessary, by this definition, means that God has always sustained it.

I would argue, however, that CC is deficient in some ways in its addressing of the aforementioned objections. Consider the first dilemma related to the eternality of created objects. For much of the Christian tradition, creation has been viewed as not only involving God's continual sustenance of the creation but also including his bringing forth creation from nothing *(ex nihilo*). It is difficult to imagine a created object that was not brought into being in time. It appears, in Howell and Bradley's model, that this seemingly indispensable part of creation is absent. For if there were a point at which these objects were brought into being, it would follow that there was a point at which they did not have any ontic status, and thus they would not be eternal. Viewing abstract objects as created prevents us from being forced to say that they exist *a se* (that is of and from themselves). The aseity of a created object would clearly be problematic theologically since only God exists *a se*. While rescuing us from the aseity dilemma, however, the CC model loses a part of what Christians have meant by "creation."

It is apparent that the above observation with respect to traditional interpretations of the word "creation" is not a problem in the mind of the continuous creationist. It might be argued by those that favor CC, that time itself was created along with the spatial universe. If the creation of abstract objects took place apart from the creation of space-time, then the CC model avoids the issue regarding a point in time at which these objects were brought into being. Mathematical objects can be brought into being *ex nihilo,* outside of the creation of space-time. This would mean that God created abstract objects separately from the material universe. Despite this possibility, the language in Col 1:16 that states that the Son is "before all things," seems to indicate though that "things" were created in time. It could be further argued by the continuous creationist, that this language is phenomenological. Thus, while I find the passage in Col. 1 and traditional interpretation of the word creation compelling, it is not unreasonable to believe otherwise. Therefore, a retort to the CC model with respect to the first dilemma is not straightforward.

The CC model does not fare as well when the second objection regarding necessity and freedom is brought into the discussion. The church's view has been rather consistent in viewing creation as not necessary.

> We believe that God created the world according to his wisdom. It is not the product of any necessity whatever, nor of blind fate or chance. We believe that it proceeds from God's free will; he wanted to make his creatures share in his being, wisdom and goodness: "For you created all things, and by your will they existed and were created." Therefore the Psalmist exclaims: "O LORD, how manifold are your works! In wisdom you have made them all"; and "The LORD is good to all, and his compassion is over all that he has made." (Ca Pt 1,s 2, Ch 1, a 1, par 4, 295)

So it is not the case that creation is necessary, according to the Catholic catechism. The catechism quotes Revelation 4:11, in which John says that all things have been created not by necessity, but *by the will of God* . In *Summa Theologica*, Aquinas says a great deal about necessity and creation. His conclusion is consistent with the aforementioned quotation:

> Since, then, the Divine Being is undetermined, and contains in Himself the full perfection of being, it cannot be that He acts by a necessity of His nature... He does not, therefore, act by a necessity of His nature, but determined effects proceed from His own infinite perfection according to the determination of His will and intellect. (Aq Pt 1, Q 19, a 4)

The principle is that God's works, and thus creation, arise not from necessity, but from God's free decision to manifest his glory. On this basis, Aquinas argues that creation cannot result from necessity, and that it therefore cannot be eternal either. In speaking specifically about creation, Aquinas reasons:

> Nothing except God can be eternal. And this statement is far from impossible to uphold: for it has been shown above (Question [19], Article [4]) that the will of God is the cause of things. Therefore things are necessary, according as it is necessary for God to will them, since the necessity of the effect depends on the necessity of the cause (Metaph. v, text 6). Now it was shown above (Question [19], Article [3]), that, absolutely speaking, it is not necessary that God should will anything except Himself. It is not therefore necessary for God to will that the world should always exist; but the world exists forasmuch as God wills it to exist, since the being of the world depends on the will of God, as on its cause. It is not therefore necessary for the world to be always; and hence it cannot be proved by demonstration. (Aq Pt 1, Q 46, a 1)

So I would suggest that neither scripture nor the church's intellectual history supports the necessity and eternality of creation. Thus, either mathematical objects are created and thus not eternal and necessary, or they somehow exist eternally apart from creation. Aquinas seems to think that the only thing that exists eternally is God himself.

Augustine, in offering an alternative, would suggest that mathematical objects exist as part of God's Wisdom. God's Wisdom is necessarily eternal and part of his being. This would give mathematical objects an ontology outside of creation, avoiding the above problems. We see evidence for such a view of mathematics in the personification of Wisdom found in Proverbs 8. “Many early Christian exegetes saw this text as pointing to the divine Logos [Word], 'the true Wisdom'” (Ha 65). Wisdom is clearly described as in use in God's creative work; we see then that Proverbs 8 is also referring to Christ. The language used in the proverb to describe the ontic status of Wisdom is like that used to describe the begotten Son. God is “that immutable truth, comprising everything that is immutably true” (Au 65). As Christ is not part of the creation, neither is Wisdom part of the creation. Rather Wisdom is God's nature.

But how do numbers (and mathematical objects more generally) relate to God's Wisdom? If mathematics is, in fact, the language the universe speaks, then the “craftsman” at God's side in creation must, as I have mentioned, be mathematical in nature. Augustine wrestles quite a bit with this relationship. While Augustine would characterize Wisdom as “far more worthy than number,” he goes on to say that “number and Wisdom are somehow one and the same thing” (Au 63,64). Later, however, he asserts that Wisdom and number “are the same” (Au 65). One of the most profound observations made by Augustine is his use of brightness and heat as a comparison to suggest that Wisdom and number exist consubstantially and cannot be separated. In the final analysis, he concludes that, “it is clear that both [Wisdom and number] are true, and immutably true” (Au 65). As quoted above then, God's nature includes number as immutably true. Therefore God's nature is mathematical. Mathematical objects are thus necessary, eternal and uncreated, according to Augustine.

Before continuing with a detailed mathematical description of the ontology of number, there is another matter relating to number's eternality that is worthy of mention. It could be argued that the Trinity's eternal ontic status guarantees like status for number. That is, it is difficult to imagine a moment at which God existed as three in one, yet neither three nor one had being. We must tread very carefully in this area as with any consideration of the Trinity for heresy is a trap easily fallen into. The property of threeness, as considered mathematically, is different in some rather critical ways from the way in which threeness is manifested in the Trinity. In particular, a property essential to the integers is that of order. That is, one and only one of the following is true: $x<y$, $x>y$ or $x=y$ for any integers x and y. This fact about integers does not apply to the Trinity, for the three persons in the Godhead are distinct yet consubstantial. Conversely, if we hold to a set theoretic definition of threeness, the elements of the set with cardinality three are distinct. There is no sense in which the elements share identity. Again, this language is not appropriate in a description of the mystery of the Trinity. Although something true about God's nature is communicated through the mathematical notions of three and one, as with all metaphors related to the Trinity, they are insufficient in communicating many essential aspects. I am hesitant at best therefore, as a mathematician, to draw any conclusions about the ontology of number from the Trinity.

The Nature of Mathematical Objects

Let me now continue with my ontological considerations. A summary of the recent history of mathematical philosophy, and the resulting mathematical definitions related to number will help set the stage for the postmodern considerations to be addressed later. In summarizing mathematical development, I will focus on the set of natural numbers **N**, in particular, which are foundational[2]. Gottlob Frege is rightfully

2 See (GV) for a somewhat detailed description of the group of integers as well as the real, rational and complex fields.

credited with providing the framework for rigorous descriptions of **N**, and its arithmetic. This framework has been titled *logicism*. He sought to show that mathematics was an extension of logic. So that we might understand his description, let me first state several definitions, most of which I will borrow from (GV). "The result of removing a name from a declarative statement is a predicative expression that refers to a *concept*; concepts are just what predicative expressions denote" (GV 20). So by removing the name "Chris" from the statement:

Chris has taken a course in dynamical systems.

we arrive at a concept under which some objects fall and others do not. That is, the objects for which the predicate:

______ has taken a course in dynamical systems.

is true fall under the concept to which the predicate refers. Now, the collection of objects that fall under a concept *F* is the concept's *extension*. Frege claimed that every concept has an extension.

We are now ready to consider Frege's definition of number. We say that two concepts *F* and *G* are *equinumerous* if there exists a one-to-one correspondence between *F* and *G*. "That is, there exists a relation *R*, such that every object that falls under *F* is related by *R* to a unique object that falls under *G*, and vice versa" (GV 29)[3]. The *number of Fs* is the extension of the concept: _____ is equinumerous with *F*. This definition places all concepts that are equinumerous into clusters. All concepts under which exactly *n* objects fall, are collapsed into one collection. This collecting defines a

3 In symbolic notation, this is equivalent to:
$$\exists R\,[\forall x\,(Fx \rightarrow \exists_1 y\,(Rxy \wedge Gy)) \wedge \forall y\,(Gy \rightarrow \exists_1 x\,(Rxy \wedge Fx))]$$

relation[4] between concepts. This relation partitions all concepts into classes (identical to the aforementioned collections). Therefore, a *cardinal number* is a class as defined by the above relation. So we say that *n* is a cardinal number if and only if there exists a concept *F* such that

n=the number of *F*s.

0 is thus, for example, the number of the concept: ___ is not self-identical. Everything is identical to itself, so nothing falls under this concept. Frege then defines the successor relation which is roughly "one bigger than."[5] Satisfying the successor relation for 0 is 1, defined to be the number of the concept: ___is identical to 0. One object, 0 alone, falls under this concept. Similarly, 2 is the number of the concept: ___ is identical to 0 or 1. We then continue *ad infinitum*.

To define the natural numbers, we consider the concepts *F* such that:

i) 0 falls under *F*, and

ii) whenever *x* falls under *F*, the successor of *x* also falls under *F*. That is *F* is *hereditary with respect to successor*.

We say that *k* is a *natural number* if and only if *k* falls under every concept *F* such that *F* satisfies both *i*) and *ii*). So why is 3, for example, a natural number? Let *F* be an arbitrary concept satisfying *i)* and *ii)*. Since *F* satisfies *i)*, 0 falls under *F*. Since *F* satisfies *ii)*, the successor of 0, namely 1, falls under *F*. Similarly, 1's successor, namely 2 falls under *F*, and thus 2's successor, namely 3 falls under *F*. Since the selection of *F*

4 Such relations are called equivalence relations in mathematics. Equivalence relations are relations defined on sets such that every element *a*, *b*, and *c* satisfy the following three properties:

a. Reflexive: *a* is related to itself.

b. Symmetric: if *a* is related to *b*, then *b* is related to *a*.

c. Transitive: if *a* is related to *b* and *b* is related to *c*, the *a* is related to *c*.

It is not difficult to see how Frege's definition of number is an equivalence relation.

5 See (GV 20) for a rigorous definition of successor that is not circular.

was arbitrary, 3 falls under all such *F*'s. So 3 is a natural number.

The definition of **N** involves quantifiers ranging over both objects and concepts. The ability to exclude objects not in the naturals depends on this second order quantification. Borrowing again from (GV), let us consider why Julius Caesar is not a natural number. Julius Caesar does not fall under every concept under which 0 falls and which is hereditary with respect to successor. Under which concept satisfying the needed properties does Julius Caesar fail to fall? He fails to fall under the concept natural number itself. The natural numbers are a concept satisfying both *i*) and *ii*), but Julius Caesar does not fall under this concept.

> There is no circularity here: The predicate “natural number” is not used in the definition. Rather, the concept it denotes must be reckoned in the domain of the concept's second order quantifier. (GV 35)

So natural number is one of the concepts included in *F*'s range in the definition of natural number. When a quantifier in a definition includes the object being defined we say that the definition is *impredicative.*

In the early 20th century, Bertrand Russell discovered a contradiction in Frege's model of allowing impredicative definitions. In 1902, he informed Frege of the contradiction, which became known as Russell's Paradox, just before Frege was to publish volume two of his *The Basic Laws of Arithmetic*. To understand the nature of the paradox, consider the following sets:

$F=\{x \mid x \text{ is a finite set}\}$, and

$I=\{x \mid x \text{ is an infinite set}\}$.

So the set of human beings is in *F*, not in *I*, whereas **N** is in *I*, not in *F*. Frege's error was in assuming that all concepts have extensions. Let us consider the concept:

___ is a set that is not a member of itself,

with the extension being the set of all sets that are not members of themselves:

$R=\{x \mid x \text{ is a set and } x \notin x\}$.

So, for example $I \in I$, thus $I \notin R$. That is, since *I* is an infinite set, it belongs to the set of all infinite sets and is thus a member of itself. Therefore *I* does not belong to *R*. We are now prepared to bring the paradox into view. If we suppose that $R \in R$, then *R* is a set such that $R \notin R$, which is a contradiction. Similarly, if we suppose that $R \notin R$, then since *R* is a set, $R \in R$, another contradiction. So it is the case that *R* is neither in *R*, nor not in *R*. This is Russell's Paradox.

To address this dilemma, Russell and Whitehead, in their *Principia Mathematica,* developed the ramified theory of types. The simple theory of types places all objects in a hierarchy of types. The lowest level is comprised of objects that are not sets. The next level is made up of sets of elements from the lowest level. Higher levels are made of sets of sets from the level immediately below. (GV) articulate the simple theory of types succinctly:

> ...if *x* ranges over all individuals, the set *A* will be the set of individuals; if *x* ranges over sets of individuals, the set *A* ranges over sets of sets of individuals; Thus *A* is not a possible value of *x*, so *A* cannot be considered for elementhood in itself. (GV 46)

Therefore, the set *R* in Russell's Paradox cannot be formed.

The ramified theory of types is more subtle and is beyond the scope of this paper. The above discussion, though, provides the central idea, which is a ban on impredicative definitions. Russell and Whitehead's description of number in *Principia Mathematica* is quite similar to Frege's except for the aforementioned prohibition. This ban, however, is not without cost. The prohibition forces Russell and Whitehead to include an Axiom of Infinity, asserting that there are infinitely many objects from which sets can be formed. Most consider this a fatal blow to the project of reducing all mathematics to principles of logic for it is not at all clear that the Axiom of Infinity is evident to all. Despite this deficiency though, Frege and Russell have been foundational in providing an axiomatic foundation to number and arithmetic.

The standard model for mathematicians today is the axiomatic theory entitled the Zermelo-Fraenkel theory of sets. Named after German mathematicians Ernst Zermelo (1871-1953) and Abraham Fraenkel (1891-1965), the theory is referred to as ZFC. C stands for the somewhat controversial Axiom of Choice. ZFC borrows a great deal from the foundational ideas of Frege and Russell as well as the work done by 19th century mathematician Georg Cantor. The only ontological assumptions made are regarding the existence of sets. ZFC supposes that even the elements of sets are sets. The existence of the empty set, ϕ (the set containing no elements), allows us to form other sets without an infinite regress. For example, $\{\phi\}$ is a set that is distinct from ϕ. Since the empty set has no elements we are not able to regress past ϕ. While ZFC makes no claim to be a branch of pure logic, it does allow impredicative definitions. The paradoxes from which these allowances arise are avoided by an additional axiom: The Axiom of Comprehension. This axiom would not allow for the set:

$$R=\{x \mid x \text{ is a set and } x \notin x\},$$

but it would allow the formation of, for any set *A:*

$$R_A =\{x \in \mathrm{A} \mid x \text{ is a set and } x \notin x\}.$$

This avoids Russell's paradox.

ZFC then defines:

$0=\phi$,

$1=\{0\}=\{\phi\}$,

$2=\{0,1\}=\{\phi, \{\phi\}\}$,...

$n=\{0,1,2,...,n-1\}=\{\phi, \{\phi\},\{\phi,\{\phi\}\},...\}$.

Thus, even the elements of sets are themselves sets. This definition also provides the appropriate definition for the successor m, of a natural number n to be $m = n \cup \{n\}$. To actually define the set of natural numbers, we need the Axiom of Infinity, which asserts **N**'s existence and that it is precisely the set of natural numbers.

Postmodern Considerations

ZFC provides a strong axiomatic foundation for mathematics. It comes at a price, however. Given many of the axioms in ZFC which avoid paradox, the system is unable to describe mathematical objects in purely logical terms. This was precisely the assertion of L. E. J. Brouwer. He argued that ZFC's axioms lacked sufficient generality to be considered a reduction of mathematics to logic. His program, which he termed *intuitionism*, sought to construct mathematical objects via a finite number of processes only. The Axiom of Infinity, for example, is rejected in the intuitionistic school.

Also important in the history was *formalism* or *finitism* which was championed by David Hilbert (1862-1943) in the first half of the twentieth century. Hilbert's aim was primarily epistemological (although mathematical epistemology and ontology cannot be separated). He sought to place mathematics on a firm epistemic foundation by proving its logical consistency. Hilbert felt that this would wed the logistic and intuitionistic approaches by constructing mathematics with an uninterpreted, symbolic, purely formal, yet consistent and epistemologically certain system. Here mathematics was nothing more than a formal interaction of symbols and syntactic rules. Mathematical statements were true only in a formal sense. Hilbert's project was doomed, however, as a result of Gödel's two incompleteness theorems which placed serious limitations on the formalist's hopes of epistemic certainty. [6]

Therefore, it could be argued that, since the foundational axioms of ZFC are not reducible to strictly logical concepts, all claims of establishing mathematical truth on absolutely certain logical foundations are groundless. This is the assertion of a relatively recent postmodern branch of mathematical philosophy known as *fallibilism.* The fallabilist critiques not only attack absolutist assumptions, but also attack claims that all proofs currently accepted as true by the mathematics community are completely rigorous.

Significant thinking in this area has been done by Paul Ernest, Reader of Mathematics Education at the University of Exeter. Drawing on the thinking of Lakatos and some interpretations of Wittgenstein, Ernest's fallabilist epistemic assumptions lead to mathematical ontologies quite different from Platonism. Wittgenstein held that logicism's view of necessity was flawed. Rather than holding that mathematical truths were necessary, following deductive paths from self-evident axioms, he held that mathematical reasoning was the following of rules inherent in language games and forms of life. Ernest writes of Wittgenstein:

> His view is that necessity, such as that of an inference following the laws of deductive logic, arises from the human agreement in following a rule that is stipulated by (and presupposed by and embedded in) a language game. (Er 71)

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6 For a more complete treatment of Gödel's incompleteness theorems, see *On Formally Undecidable Propositions of Principia Mathematica and Other Systems* by Kurt Gödel, or (GV).

Thus it is agreement (that the player is playing according to the rules) that takes the place of logical necessity. These rules are defined socially. "Terms and sentences do not in general have distinct individual references" (Er 70). They are shared culturally, within the mathematics community, for example. Rational "certainty" then becomes victory in argument. Ernest writes,

> Ultimately a proof is a narrative for human consumption, a "procedure that is plain to view," not a superhuman objective structure. For the primary function of a proof is to convince, and logical structure is a means to that end. Thus mathematical knowledge is founded on human persuasion and acceptance. (Er 83)

This is quite distinct from a Platonic model in which the mathematician is describing necessary mathematical objects based on equally necessary and universal rules of deductive logic, drawing conclusions from self-evident axioms.

> There is no extrahuman or objective force that compels anyone to follow a logical rule or to accept the conclusion of a logical deduction. It is, rather, that participating in certain language games entails accepting certain rules. If one rejects the rule, one is repudiating the game as it is understood and played by others. (Er 71)

This is quite a revolutionary view of deductive logic that is not without its detractors. Michal Dummett writes that Wittgenstein "goes in for full blooded conventionalism; for him, the logical necessity of any statement is always the direct expression of a linguistic convention" (Du 495). In Dummett's interpretation of Wittgenstein, when considering the proof of a theorem for example, "we could have rejected the proof without having done any more violence to our concepts than is done by accepting it; in rejecting it we could have remained equally faithful to the concepts with which we started out" (Du 497). Dummett would reject Wittgenstein's model as he finds it unimaginable that one, not for lack of understanding, would reject the proof, while another would accept it.

Ernest disagrees with Dummett's reading. He calls Dummett's interpretation “a parody based on a misunderstanding” (Er 75). In Ernest's reading of Wittgenstein's theory of mathematical language games, the model differs from naive conventionalism in that one is not accepting these conventions arbitrarily. Rather they are woven into socially shared practices of language and forms of life. Thus, it is important to note that Ernest is not suggesting that the rules are in any way subjective. To shed light on Ernest's perspective, it is helpful to understand how he defines objectivity. For Ernest, objectivity does not find its meaning in some ethereal reality. Instead, objectivity is found via acceptance within a community of shared forms of life. The sense in which mathematical knowledge is objective is social. Mathematical “truth” does not change on the whim of the subject. It is external, “thing-like,” but not eternal or ideal. Knowledge is that which is shared; it is intersubjective, being formed within the mathematics community (Er 145). Wittgenstein himself writes

> And yet we can say: the person who is calculating in my language game does not think of it as particularly of his nature that he gets this; the fact does not appear to him as a psychological one... He says not, “so that's how I went,” but, “so that's how it goes.” (Wi 477)

The one doing the calculating is thus not free to make things up as she goes. In this sense she is compelled by something outside of herself. The difference between objectivity in the Platonist's mind, and objectivity as Ernest sees it, lies in that this compulsion is not extra-human; rather it arises from her need to follow normative linguistic practices. So proof acceptance is indeed a decision, but it is not made by the individual; it is made by community. The rules themselves are arbitrary, however. This postmodern understanding is supported by Wittgenstein when he writes in his *Remarks on the Foundations of Mathematics* that, “not only the axioms but the whole of syntax is arbitrary” (p. 104).

In addition, as Ernest interprets Wittgenstein's philosophy, the rules of the language game are open to change. “Forms of life may develop and change. Similarly language games have an open texture and may grow, change and lead in unanticipated directions” (Er 71). Therefore, mathematical “truths” are not necessarily eternal. The temporality of mathematical objects follows from their existence merely as social constructs. Just as the rules of the language game are subject to change as the mathematical community evolves and is affected by culture and its own history, so too can the mathematical “objects” change. In a clear attack on Platonic ontology, Ernest quotes Wittgenstein:

> It is already mathematical alchemy, that mathematical propositions are regarded as statements about mathematical objects – and mathematics as the exploration of these objects?...All I can do, is to show an easy escape from this obscurity and this glitter of concepts.. .which only looks like a gleam without a corporeal substrate when seen from ... [an] other direction. (Er 76)

This is the foundation upon which Ernest's philosophy of mathematics, known as *social constructivism,* is built. Besides Wittgenstein, another important influence on Ernest's social constructivism is the philosophy of Lakatos. Ernest argues that Lakatos, using ideas from Hegel, presents a fallabilistic epistemology of mathematics that uses dialectic as its logic. This influence places the role of history as critical next to Wittgenstein's language games in Ernest's ontolgy. Hegel's dialectic “begin[s] with a proposition, move[s] to a contradiction/refutation of it, followed by a new proposition... Hegel's dialectical triad provides a model for conceptual continuity and change in the growth of mathematical knowledge, both logically and historically” (Er 106). Hegel's thinking evolves into Lakatos' Logic of Mathematical Discovery (LMD), the details of which are beyond the scope of this document. What is important for our purposes is to understand the way in which Lakatos fits into Ernest's model of social constructivism. That is, the place of history and dialectic in the evolution of mathematical knowledge, along with Wittgenstein's thinking, is the basis of social constructivism.

> ...a key strength of Lakatos' philosophy of mathematics is that it is not prescriptive but descriptive and that it attempts to describe mathematical practice and mathematics as it is historically, not as it ought to be practiced or reorganized in order to fulfill some foundationalist plan. (Er 126)

So Hegelian dialectic replaces absolutist certainty based on deductive logic[7].

The way in which dialectic takes place in mathematics is through the written narrative of proofs. It would appear that written proofs are not by nature dialogical (that is, using dialectical logic), but monological. But, as mentioned before, to the constructivist, proofs serve to convince, rather than provide a platform for certainty. So the proof begins the dialectical process associated with Lakatos' LMD in which refinement and counterexample are central. According to Ernest, the dialogical nature of mathematics has been suppressed by absolutist ideology. This suppression denies the role of the social context and of history in the process of mathematical discovery, as mathematics is being abstracted and formalized.

From my perspective, Ernest provides some compelling evidence to support this position. If, in fact, a proof serves to convince, then “a listener/reader is presupposed” (Er 169). More compelling is the way in which dialectic can be seen in mathematical concepts themselves. Consider the definition of the convergence of a real

7 It should be noted that, as Ernest admits, some editors of Lakatos have claimed that late in life, Lakatos “rejected” his fallabilist position. Also, Lakatos himself purged all reference to Hegel in his later work. Ernest's position, however, is that, despite this rejection of Hegel, the influence of Hegelian synthesis on Lakatos' writing is clear, and the evidence that he abandoned his fallabilist position is weak.

valued sequence: A sequence of real numbers a_n is said to converge to a real number *a* if, for each $\epsilon > 0$ that is given, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $|a_n - a| < \epsilon$. When I teach first semester Real Analysis to my students, I approach this concept from a dialectical viewpoint. One who calls into question the convergence of a particular sequence is called upon to provide a small enough ϵ so that, regardless of the n_0 chosen by the person asserting convergence, the absolute value condition fails to be met. The concept is conversational in that it sets up a challenge/response paradigm. In addition, the refereeing process, which is the means by which new theorems are accepted into the mathematical community, can be seen as dialectical in nature. We see here some validity in Ernest's assertion that the human element of the evolution of mathematical knowledge cannot be ignored.

Ernest defends social constructivism from charges of full fledged, “anything goes” relativism. He admits that constructivism is indeed relativistic, but he distinguishes forms of relativism in which choices are made arbitrarily (strong relativism) from a form of relativism in which necessity and stability of mathematical knowledge results. Ernest cites three tenets of “strong relativism” with which he disagrees (Er 252):

1. Knowledge must be relativized to each individual knower.
2. We know nothing about the world or its existence.
3. We must make an arbitrary decision whether to accept or reject every putative item of knowledge.

Ernest says that the rules of the language game within the mathematical community stand against the first and third tenets. Regarding the second tenet, he counts that knowledge of the rules of the language game, the associated forms of life and the socio-historical setting of mathematics as evidence for some knowledge about the world.

On the other hand, Ernest also denies the extreme of absolutism. He rejects three central absolutist theses as well (Er 249):

1. Universalism: All people at all times in all cultures could be brought to agree on the assessment of meaning, truth and existence.
2. Objectivism: The assessment of meaning, truth and existence and their foundations can be presented so that they are independent of the point of view of persons, cultures and humankind.
3. Foundationalism: There is a unique and permanent foundation for all assessments of meaning, truth and existence.

He says of these:

> Social constructivism denies all three of these with respect to both mathematical knowledge and the ontology of mathematics, but this denial is because the claims are too strong, not because anything goes. (Er 250)

The first thesis places the knowing subject outside of the socio-historical influence central to social constructivism. Ernest argues that gaining mathematical knowledge requires a person "to be educated over many years in school, and ... to learn to successfully participate in mathematical language games and forms of life" (Er 250). Thesis two also suggests that the epistemic agent is able to transcend the "human context" in which he finds himself, which is thought to be impossible by the social constructivist. Finally, as mathematical objects are neither eternal nor necessary conclusions of infallible deductive arguments, the social constructivist rejects thesis three. So Ernest categorizes his philosophy of mathematics not as absolutist, but as relativist, while keeping it distinct from strong relativism.

A Christian Consideration of Social Constructivism

I deliberately use an indefinite article in the title for this section, for I hold Howell and Bradley's assessment in high regard and wish to place my own comments alongside theirs. There are several ways in which Ernest's social constructivism is consistent with the Christian world view.

First, the fallenness of mankind and the resulting potential for error in employing reason seems to fit nicely with some fallabilist assumptions. Consider chapters 38 through 41 of Job in which God says, among other things:

> Then the LORD answered Job out of the storm. He said: "Who is this that darkens my counsel with words without knowledge? Brace yourself like a man; I will question you and you shall answer me. Where were you when I laid the earth's foundation? Tell me if you understand. Who marked off its dimensions? Surely you know. Who stretched a measuring line across it?" (Job 38:1-5)

This passage certainly gives me pause before making haughty claims regarding my absolute knowledge of mathematical objects in the mind of God. As is the case with all forms of knowledge, a fallen nature is prone to error when discerning mathematical knowledge. The Christian must therefore be ever diligent in his thinking to ensure that his description of mathematical objects is as accurate as possible. As is the case with all hermeneutics, any reading of these objects can be flawed and is open to critical analysis. In particular, in a ZFC axiomatic system, the foundational axioms themselves involve assumptions that must be filtered through Scripture and rigorous criticism.

A second component where I see similarity between Ernest's thought and Christianity is in the area of written dialectic. Like the written narrative of proof, God's Word also presupposes a reader. In addition to his special revelation, the cosmos speaks to us. Consider Romans 1:19-20, where the apostle Paul writes,

> ...since what may be known about God is plain to them [mankind], because God has made it plain to them. For since the creation of the world God's invisible qualities – his eternal power and divine nature – have been clearly seen, being understood from what has been made, so that men are without excuse.

Mathematical objects themselves, in my ontology, are not among those things that have been made (since they exist as God's eternal Wisdom and thus as his nature), but the creation speaks the language of mathematics, revealing God's character. We saw in Job a sort of dialectic between Job and the Lord. Isaiah 1:18 reads, "Come let us reason together, says the LORD." We must be careful here, though, for the nature of the dialectic between God and man is different from dialectic as it has been viewed traditionally in much philosophy. Classically, dialectic involves two parties on rather equal footing, engaging in logical dialog so that they might be "distinguishing truth from error" (Ho 15). However, the sort of dialectic that takes place between Job and God, and that spoken of in Isaiah, is dialectically one sided. It is God allowing himself to be known. Humanity has nothing to contribute. As God communicates with us through his Holy Spirit, we may play the role of inquirer, but we do nothing to influence what is eternally true. This one-sided dialectic is also distinct from the dialectic envisioned in mathematics by Ernest. In Lakatos' LMD, a "proof" (which in LMD is a "rough thought-experiment or argument, decomposing the primitive conjecture into subconjectures or lemmas" (Er 112)) is continually open to refinement and counterexample. This divine dialectic indeed has a pedagogical character, but human refutation and refinement of God's truth is unacceptable to the Christian. On the other hand, *understanding* of that truth can and should be refined, as Christians continually run their understanding through the filter of Scripture and what is observed in the cosmos. It is in this sense that Christian epistemology is dialectical. Models of knowledge are continually being refined as believers, with the help of the Holy Spirit, listen to God's side of the dialog through his Word and generally through the cosmos.

Despite these similarities, I would argue that there are also some points in which social constructivism and a Christianity that takes the Bible seriously clearly cannot be reconciled. That thoughts exist and are eternal in the mind of God is central to Christianity. While knowledge of Goid's truth may be fallible, being strongly influenced by history, culture, etc., he is the unique and permanent foundation for all assessments of meaning, truth and existence. As I hold to the Platonic/Augustinian ontology of number, mathematical objects also, in my view, exist eternally outside of us. Thus, these objects are not subject to change, for God's Wisdom is immutable.

What perhaps is more central is not the existence of the objects (for Ernest would describe both the objects and the knowledge about them as socially constructed, and in that sense, objective), but rather the nature of our access to these objects. To Ernest, access is not an issue since the mathematician (or the mathematical community) is building them. They are, by nature, knowable. In the Platonic/Augustinian ontology described above, the appropriate question arises as to how we are able to know these ethereal truths that are not oriented in space-time. I think Christianity offers a compelling answer. To the Christian, we are all created in God's image. We are thus endowed with the ability to reason and use logic. So we are partakers in God's Wisdom and the eternal mathematical truths that lie therein. In Proverbs 8, which is central to this ontology, Wisdom herself implores us to enter into her sweet communion:

> Does not Wisdom call out? Does not understanding raise her voice? On the heights along the way, where the paths meet, she takes her stand; beside the gates leading into the city, at

> the entrances, she cries aloud: "to you, O men, I call out; I raise my voice to all mankind. You who are simple, gain prudence; you who are foolish gain understanding... For whoever finds me finds life and receives favor from the LORD." (Proverbs 8:1-5, 35)

There is evidence in verse 35 that Wisdom is also a metaphor for the Christ. The beauty of the metaphor deepens in that it is through Christ that ultimate accessibility to God's Wisdom is possible. (Consider I Cor. 1:18-25). It is Wisdom's accessibility that is noteworthy in our current discussion. She calls us the gain understanding; it is her desire that we do so.

I acknowledge that, given our falleness, we are unable to participate fully in this communion. It is evident that the power of the gospel gives us hope. Regardless of the incompleteness of our participation, participate we can; participate we must. Indeed the Christian can learn something from social constructivism. Both the Christian and the social constructivist, find themselves outside of complete certainty, engaging in dialectic through history. One critical difference lies in the fact that, to the Christian Platonist, dialectic potentially moves our knowledge toward the object in question as we become conformed to Christ's likeness. We have good reason to believe that mathematical truth is attainable.

I would also argue that, given Christian presuppositions, we have good reason to believe in the reliability of deductive logic, and therefore in the theorems that follow deductively from the foundational axioms in ZFC. To understand my rationale, recall that, to the Christian Platonist, there exists eternal mathematical objects in the mind of God that are accessible because of God's image in us. Since this is God's image, the reasoning that follows, using deductive logic, is a reliable avenue to truth. By contrast, Ernest would suggest that mathematical objects are socially constructed and that the deductive logic I see as following from God's image in us is really a language game. Deductive logic being such, we cannot hope that it leads us to anything that is eternally true. Mathematical truth is thus open to change as language games and the socio-historical context of the mathematical community evolves. From the standpoint of the social constructivist, it would be wrong to expect that different members of the current mathematical community would arrive at different “mathematical truths” since, for the most part, members of this community are all following the rules of the same language game. However, given Ernest's ontology, we might expect that cultures doing mathematics in independent spheres would arrive at differing mathematical conclusions.

This does not seem to be the case however. Western, and to a lesser extent Islamic mathematics, has been greatly influenced by Greek thought. It could be argued, therefore, that the uniformity of mathematics that is to be found among these two cultural traditions is due to the adoption of Greek mathematical language games. The mathematics done in the ancient far east, on the other hand, should be above such criticism as it developed in isolation from Hellenistic influence. Yet, when we investigate Chinese mathematics from the Shang period (about 1200 B.C.) forward, we find several significant parallels with developments in the west. Among them are Liu Hui's discovery of the Pythagorean relationships among the sides of right triangles. Another is the generalization of “the method of extracting of square roots to equations of higher degree leading to the numerical method of solving algebraic equations we today refer to as Horner's method” (Ev 216) (named for British mathematician William Horner, who lived from 1786-1837). There exists evidence that the binomial theorem and Pascal's triangle were known in China before the 1300's B.C. Some of the most interesting results from the region involve extremely precise approximations of pi.

We see here discoveries that are arithmetic, geometric, analytic and algebraic in nature. Despite the fact that Chinese methodology was different[8], the mathematical objects and relationships proposed demonstrate tremendous likeness to those found in the west. It seems as if something has led persons from vastly different world views and cultures to the same truths. I submit that it is deductive logic, available to all via the divine image in us, that has led these distinct peoples to eternal truths. If Ernest's ontology is correct, we would expect to see some variance in the nature of these discoveries, when in fact we see very little. While similarities across cultures are by no means proof of a Christian Platonic ontology of mathematics, it does provide good evidence for such belief.

Further evidence exists in Wigner's characterization of the “unreasonable effectiveness” of mathematics in describing the physical universe. As I have mentioned, the assertion that mathematics is the most effective avenue for describing the universe has been held by thinkers from ancient Greece to Galileo to Gödel. Some critics would instead argue that the laws of nature are mathematical precisely because we choose to describe them as such. It is a cultural phenomenon. But, as mathematical physicist Paul Davies points out,

> ... I confess that this claim [that the perceived relationship between mathematics and the universe is a cultural phenomenon] is altogether too glib, for a number of reasons... Much of the mathematics that is so spectacularly effective in physical theory was worked out as an abstract exercise by pure mathematicians long before it was applied to the real world. The original investigations were entirely unconnected with their eventual application. (DA 151)

Davies as well as most scientists holds to the notion that “the major advances in

8 Much of Chinese mathematics appears to be inductive rather that deductive, moving from specific examples to general principles. The goals were pedagogical, rather than to address skepticism. Proofs were often pictorial. See (HB 56-61) and (Ev 211-219) for more detail about China's rich mathematical history.

mathematical physics really do represent discoveries of some genuine aspect of reality, and not just reorganization of data in a form more suitable for human intellectual digestion" (Da 152). On the other hand, it could be argued that the human mind has simply evolved in an arbitrary way that just happens to correspond to the physical world. But it seems that evolution dictates that such an evolution is *not* arbitrary. If the universe is not to be described mathematically, then the fittest organism would be one that describes the universe otherwise. Putting it another way, if reality can be described mathematically, then the fittest organism would be the one whose brain evolved with the faculties to understand the universe accordingly. This supports the idea that our mathematical inclination reflects something true about the cosmos. With regard to the thought that this evolution is taking place arbitrarily, Davies points out that "it is very hard [then] to see why abstract mathematics has any survival value" (Da 152). That is, if our minds have evolved with mathematical competence for the sake of understanding the world better to make us more fit for survival, why has pure mathematics come to be? What survival benefit does mathematics for its own sake provide us? The connection between our understanding and love of abstract mathematics, and its effectiveness in describing the universe do not seem to be cultural nor an "an accident in our evolutionary history" (Da 151).

Alternatively, I would suggest that the mathematical nature of reality (creation), our intellectual predisposition to logic and mathematics, are teleologically designed. Quoting Roger Penrose, Davies writes "There must... be some deep underlying reason for the accord between mathematics and physics, i.e. between Plato's world and the physical world" (Da 152). It is quite coherent that the creation would demonstrate a temporal reflection of what eternally exists in the mind of God. It therefore stands to reason that with his image in us, we could accurately describe that reflection with his eternal Wisdom in mathematical form. In this model, pure mathematics exists because of our divinely given desire to uncover God's eternal Wisdom. Applied mathematics then serves to use these eternal objects to accurately describe a universe that has been created by him in whose mind these objects exist.

Many mathematicians have likewise considered these objects eternal, unchanging and beautiful. When considering the Mandelbrot set (a simply generated yet graphically complex geometric form known as a fractal in chaotic dynamical systems) Roger Penrose says,

> much more comes out of the structure than is put in in the first place. One may take the view that in such cases the mathematicians have stumbled upon the 'works of God'... It is a feeling not uncommon amongst artists, that in their greatest works they are revealing eternal truths which have some kind of prior ethereal existence... I cannot help feeling that, with mathematics, the case for believing in some kind of ethereal, eternal existence... is a good deal stronger. (Da 143,144)

Atheistic mathematician G.H. Hardy wrote that he studied mathematics for its beauty, not its application. One has good reason to believe that what we experience as mathematicians when we uncover mathematical truth is the joy of uncovering God's Wisdom. Consider Proverbs 2:10: “For Wisdom will enter your heart and and knowledge will be pleasant to your soul.”

While Wisdom in Proverbs 8 may be a description of the second person of the Trinity, mathematical objects also find their place nicely in the Logos. This affords the Christian mathematician a rather exciting vocation. When doing pure mathematics, the mathematician is studying part of God's very nature, true, necessary and eternal. Mathematical study then can clearly be seen as an act of worship. Let me be clear on this point. Study of any of the natural sciences constitutes an act of worship for the Christian scholar. Discoveries about the universe reveal characteristics about our God, allowing us in a deeper way to ascribe to him "worth-ship." We must not go too far and say that mathematics is the only discipline that can make such a claim. Nevertheless, there does seem to be something unique about mathematics as it relates to God in comparison to other natural sciences. Whereas physics, for example, studies the creation, revealing aspects about the creator, if in fact mathematics is part of God's Wisdom, then mathematical study seems in some sense to be a rather *direct* study of God's very nature.

I conclude with the following thoughts from Augustine as he considers Ecclesiastes 7:25-26:

> Men to whom God has given the ability in argument, and whom stubbornness does not lead into confusion, are forced to admit that the order and truth of numbers have nothing to do with the bodily senses, but are unchangeable and true and common to all rational beings.... Not without reason was number joined to wisdom in the Holy Scriptures where it is said, "I and my heart have gone round to know and to consider and to search out wisdom and number." (Au 56-57)

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